Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

Let $\mathcal{H}$ be a Hilbert space. Let V any closed subspace of $\mathcal{H}$; recall the definition of $V^{\perp}$ as

$$V^{\perp} := \{f \in \mathcal{H} \mid \langle g, f\rangle = 0 \ \forall g \in V\}\,. \tag{1}$$

We saw in class that the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = V \oplus V^{\perp}$, meaning that $V \cap V^{\perp} = \{0\}$ and that for any non-zero $f \in \mathcal{H}$ there exists a unique element $f_V \in V$ such that $f - f_V \in V^{\perp}$. Define $P_V f := f_V$; from the uniqueness of f_V this is a well defined linear mapping.

- **a** Prove that $P_V^2 = P_V = P_V^*$.
- **b** Use **a** to prove that P_V is bounded and if $V \neq \{0\}$ then $\|P_V\| = 1$.
- **c** Prove that if V_1 and V_2 are two closed subspaces of $\mathcal{H}$ then[1]

$$V_1 \perp V_2 \qquad \Longleftrightarrow \qquad P_{V_1} P_{V_2} = 0. \tag{2}$$

Exercise 2

Let $\phi(t)$ and $\psi(t)$ differentiable functions on the Hilbert space $\mathcal{H}$, meaning that the limit

$$\frac{d\phi}{dt}(t) := \lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h} \tag{3}$$

exists in the norm topology of $\mathcal{H}$ for each $t \in \mathbb{R}$, and similarly for $\psi(t)$.

Prove that

$$\frac{d}{dt}\langle \phi(t), \psi(t)\rangle = \langle \frac{d\phi}{dt}(t), \psi(t)\rangle + \langle \phi(t), \frac{d\psi}{dt}(t)\rangle \tag{4}$$

Exercise 3

Let $\mathcal{H}$ be a Hilbert space. Consider A and B bounded self-adjoint operators on $\mathcal{H}$. Prove that $\frac{1}{i\hbar}[A, B]$ is self adjoint.

[1] We denote with $\perp$ the condition of two subspaces of an Hilbert space $\mathcal{H}$ of being orthogonal, i.e., V_1 is orthogonal to V_2, or $V_1 \perp V_2$ if and only if for any $(f, g) \in V_1 \times V_2$ we have $\langle f, g\rangle = 0$.

Exercise 4

Consider a vector space V over $\mathbb{C}$, $A,\ B,\ C$ linear bounded operators on V and $\alpha \in \mathbb{C}$.

a Prove that $[A, B + \alpha C] = [A, B] + \alpha [A, C]$.

b Prove that $[B, A] = -[A, B]$.

c Prove that $[A, BC] = [A, B] C + B [A, C]$.

d Prove that $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$.